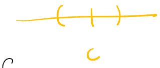
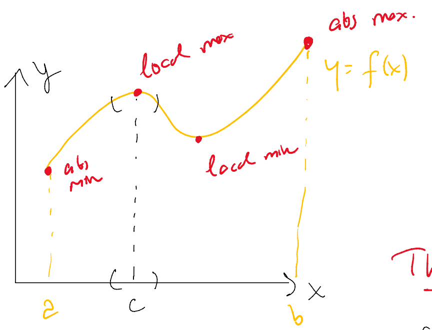
Let f be a function of a single variable and c be a point in the domain of f.

Then f is said to have



extreme values of f

- an absolute (global) maximum at c if $f(c) \geq f(x)$ for all x in the domain of f.
- ________ minimum ________ $f(c) \leq f(x)$ ________
- a local maximum at c if $f(c) \geq f(x)$ for all x in a neighborhood of c that are in the domain of f.
- ________ minimum ________ $f(c) \leq f(x)$ ________



Recall that if f is continuous on $[a,b]$, then f has an absolute max and an absolute minimum on $[a,b]$.

So how do we find these?

Theorem: Let f be a function on some interval I and suppose that f has an extreme value at c in I. Then

- c is a critical point of f, i.e. $f'(c) = 0$, or
- c is a singular point of f, i.e. $f'(c)$ does not exist, or
- c is an endpoint of the domain of f.

Proof: Suppose that c is not an endpoint of the domain of f and $f'(c)$ exists. By a theorem that we proved earlier, if f has a (local) max or min $f(c)$ and $f'(c)$ exists, then $f'(c) = 0$.

Example: If exists, find the absolute extreme values of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2,4]$.

Solution: As f is a polynomial, it is continuous everywhere, and so, on $[-2,4]$. Thus it has absolute max and min on $[-2,4]$. We have that

$$f'(x) = 6x^2 - 6x - 12 + 0 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

So we get

critical points of f: $-1, 2$ $\qquad f'(x) = 0 \Rightarrow x = 2$ or $x = -1$

singular ________ : none

endpoints of the domain: $-2, 4$

As $f(-1) = -2 - 3 + 12 + 1 = \underline{8}$, $f(2) = 16 - 12 - 24 + 1 = -19$, $f(-2) = -16 - 12 + 24 + 1 = \underline{-3}$

and $f(4) = 128 - 48 - 48 + 1 = \underline{33}$, f has its abs max at 4 and its abs min at 2.

Example: Find the abs max and min of $f(x) = 2\sqrt[3]{(x-1)^2} - x$ on $[0,4]$.

Solution: We have $f'(x) = 2 \cdot \frac{1}{3}\left((x-1)^2\right)^{-\frac{2}{3}} \cdot 2(x-1) - 1$. Then

$$f'(x) = 0 \Rightarrow \frac{2}{3} \cdot \frac{1}{(x-1)^{4/3}} \cdot 2(x-1) = 1 \Rightarrow 4\cancel{(x-1)} = 3(x-1)^{\cancel{4/3}\ \frac{1}{3}}$$

$$\frac{4}{3} = (x-1)^{\frac{1}{3}}$$

$$\Rightarrow x = \left(\frac{4}{3}\right)^3 + 1 \text{ is a critical point of } f$$

$f'(x)$ does not exist at $x = 1$, so 1 is a singular point of f

$0, 4$ are the endpoints of the domain of f

$$f\left(\left(\tfrac{4}{3}\right)^3 + 1\right) = 2\sqrt[3]{\left(\left(\tfrac{4}{3}\right)^3\right)^2} - \left(\left(\tfrac{4}{3}\right)^3 + 1\right) = \frac{32}{9} - \frac{64}{27} - 1 = \frac{96 - 64 - 27}{27} = \frac{5}{27}$$

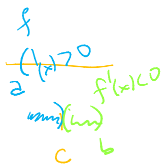
$f(1) = -1$

$f(0) = 2$ $\qquad f(4) = 2\sqrt[3]{9} - 4 > 0$

Thus f has its abs max value 2 at $x = 0$ and

—— min —— -1 at $x = 1$.

The first derivative test: Let f be a function which is continuous at c that is not an endpoint of the domain of f.



- If $f'(x) > 0$ on some interval (a, c) and $f'(x) < 0$ on some interval (c, b), then f has a local maximum at c.
- —— $f'(x) < 0$ —— $f'(x) > 0$ ——, —— minimum ——.

(There are variations of this where one assumes that c is an endpoint of the domain of f.)

Example: Find and classify all critical points of $f(x) = 3x^4 - 4x^3 - 72x^2$ on $[-5, 5]$

Solution: We have

$$f'(x) = 12x^3 - 12x^2 - 144x = 12x(x^2 - x - 12) = 12x(x-4)(x+3) = 0 \Rightarrow x = 0 \text{ or } x = 4,\ x = -3$$

and so the critical points of f in $[-5, 5]$ are $0, 4$ and -3.

x	-5		-3		0		4		5
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		$\searrow$	local min	$\nearrow$	local max	$\searrow$	local min	$\nearrow$	

f has a local min at $x = -3$ and $x = 4$.

f has a local max at $x = 0$.

Example: Find the local extreme values of $f(x) = x - \sqrt[3]{x^2}$ on $[-1,2]$.

Solution: We have that $f'(x) = 1 - \frac{1}{3}(x^2)^{-\frac{2}{3}} \cdot 2x = 1 - \frac{2x}{3x^{4/3}}$. So f has a singular point at $x=0$ and the critical point of f is $\frac{8}{27}$ since $f'(x) = 0 \Rightarrow 1 = \frac{2x}{3x^{4/3}} \Rightarrow 3\cancel{x}^{4/3} = 2\cancel{x} \Rightarrow x = \frac{8}{27}$

$x^{1/3} = \frac{2}{3}$

x	-1		0		$\frac{8}{27}$		2
$f'(x)$		$+$	$\|$	$-$	0	$+$	
$f(x)$		$\nearrow$		$\searrow$		$\nearrow$	

f has a local max at $x=0$ and $x=2$

______ min at $x = \frac{8}{27}$ and $x=-1$

What do we do if we have open intervals?

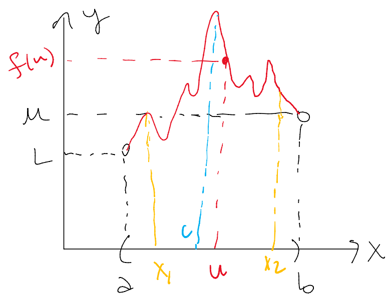
Theorem: Let f be continuous on (a,b) where $-\infty \leq a < b \leq +\infty$. Suppose that

$$\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to b^-} f(x) = M.$$

a) If, for some u in (a,b), $f(u) > L, M$, then f does have an abs max on (a,b).

b) ______ $f(u) < L, M$, ______ min ______.

Proof:



(of a): Suppose that there is u in (a,b) with $f(u) > L, M$. Then there are $a < x_1$ and $x_2 < b$ such that $x_1 < u < x_2$

$f(x) < f(u)$ for all $a < x < x_1$ and

$f(x) < f(u)$ for all $x_2 < x < b$.

As f is continuous on $[x_1, x_2]$, f does have an absolute max on $[x_1, x_2]$, say at c. Clearly $f(c) \geq f(x)$ for all x in $[x_1, x_2]$.

But we also have that $f(c) \geq f(u) > f(x)$ for all $a < x < x_1$ and $x_2 < x < b$.

Thus $f(c) > f(x)$ for all x in (a,b), so f does have an absolute max at c.

Example: Let $f(x) = x^2 + \frac{54}{x} + 5$ be defined on $(0,\infty)$. Show that f has an abs min on $(0,\infty)$ and find this value.

Solution: Clearly $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to 0^+} f(x) = +\infty$ and $f(x) < +\infty$ for all x in $(0,\infty)$.

Thus, by the theorem above, f does have an abs min on $(0,\infty)$.

We have that $f'(x) = 2x - \frac{54}{x^2} + 0 = 0 \Rightarrow 2x = \frac{54}{x^2} \Rightarrow x = 3$

$2x^3 = 54$

$x^3 = 27$

x	0		3		∞
$f'(x)$		$-$	0	$+$	
$f(x)$		$\searrow$		$\nearrow$	

As $f'(x) < 0$ on $(0,3)$ and f is cont. on $(0,3]$, $f(x) \geq f(3)$ for all $0 < x < 3$.

Suppose that $f(x) < f(3)$. Then $\frac{f(x) - f(3)}{x-3} > 0$ and by MVT $f'(c) > 0$ for some $x < c < 3$

Similarly, $f(x) \geq f(3)$ for all $3 < x < +\infty$. Thus $f(3) \leq f(x)$ for all $0 < x < +\infty$ and so f does have an abs. min at $x = 3$.